

## Number of degrees of freedom of two-dimensional turbulence

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We derive upper bounds for the number of degrees of freedom of two-dimensional Navier-Stokes turbulence freely decaying from a smooth initial vorticity field  $\omega(x, y, 0) = \omega_0$ . This number denoted by  $N$  is defined as the minimum dimension such that for  $n \geq N$ , arbitrary  $n$ -dimensional balls in phase space centered on the solution trajectory  $\omega(x, y, t)$ ; for  $t > 0$ , contract under the dynamics of the system linearized about  $\omega(x, y, t)$ . In other words,  $N$  is the minimum number of greatest Lyapunov exponents whose sum becomes negative. It is found that  $N \leq C_1 \text{Re}$  when the phase space is endowed with the energy norm and  $N \leq C_2 \text{Re}(1 + \ln \text{Re})^{1/3}$  when the phase space is endowed with the enstrophy norm. Here  $C_1$  and  $C_2$  are constant and  $\text{Re}$  is the Reynolds number defined in terms of  $\omega_0$ , the system length scale, and the viscosity  $\nu$ . The linear (or nearly linear) dependence of  $N$  on  $\text{Re}$  is consistent with the estimate for the number of active modes deduced from a recent mathematical bound for the viscous dissipation wave number. This result is in a sharp contrast to the forced case, for which well-known estimates for the Hausdorff dimension  $D_H$  of the global attractor scale highly superlinearly with  $\nu^{-1}$ . We argue that the “extra” dependence of  $D_H$  on  $\nu^{-1}$  is not an intrinsic property of the turbulent dynamics. Rather, it is a “removable artifact” brought about by the use of a time-independent forcing as a model for energy and enstrophy injection that drives the turbulence.

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### I. INTRODUCTION

Chaotic dynamics are characterized by the stretching and folding of volume elements in phase space (solution space). In the presence of dissipation, these can be accompanied by volume contraction. For a finite-dimensional system, volume elements can eventually collapse onto complex sets of zero volume having fractal structures, whose generalized dimensions, such as the box-counting and Hausdorff dimensions, are significantly lower than the phase-space dimension. For infinite-dimensional systems, volume contraction can occur for finite-dimensional volume elements. Furthermore, given a sufficiently large positive integer  $N$  (depending on physical parameters and initial conditions), this contraction can occur for arbitrarily oriented  $n$ -dimensional volume elements following a trajectory—solution “curve” in function phase space—provided that  $n \geq N$ . This is the case if the sum of the largest  $N$  Lyapunov exponents at each point of the trajectory  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ , which can possibly be different for different trajectories, is negative. The smallest  $N$  (which will be denoted by  $N$  still) satisfying this condition thus defines the minimum dimension in phase space for which all  $n$ -dimensional ( $n \geq N$ ) volume elements along a given trajectory contract during the course of evolution. This volume contraction means that the chaotic nature of the local dynamics can be “captured” and “contained” within a linear subspace having dimension not higher than  $N$ . (This subspace may continuously change along the trajectory, though its dimension does not exceed  $N$ .) For this reason,  $N$  can be thought of as an effective dimension of the dynamical system in question, in the sense that its local dynamics can be adequately described by an  $N$ -dimensional model. When an attractor (or a global attractor) exists and  $N$  is common to every trajectory having initial data containing the attractor, its box-counting and Hausdorff dimensions are both bounded from above by  $N$  [1], which is conveniently defined as the

number of degrees of freedom. More precisely, these are bounded from above by the Lyapunov dimension  $D_L$ , which satisfies  $N-1 \leq D_L < N$  and is defined by [2,3]

$$D_L = N - 1 + \frac{1}{|\lambda_N|} \sum_{i=1}^{N-1} \lambda_i. \quad (1)$$

In this study we determine upper bounds for  $N$  for two-dimensional Navier-Stokes turbulence freely decaying from a smooth initial vorticity field  $\omega_0$  in a doubly periodic domain of length scale  $L$ . Note that the global attractor for this case is trivial and has zero dimension. However, the present problem is nontrivial because it is concerned with transient dynamics, most importantly during the stage of fully developed turbulence. The bounds obtained are expressible in terms of physical parameters and found to scale linearly or almost linearly (depending on the chosen norms for the phase space) with the Reynolds number  $\text{Re}$ , which is defined in terms of  $\omega_0$ ,  $L$ , and the viscosity  $\nu$ . On the one hand, such scaling behaviors are in accord with heuristic arguments based on physical and mathematical estimates of the viscous dissipation wave number. On the other hand, these are in a sharp contrast to the forced case, for which well-known upper bounds for the Hausdorff dimension  $D_H$  of the global attractor have superlinear dependence on  $\nu^{-1}$  [4–6]. We discuss this discrepancy and argue that the superlinear dependence of  $D_H$  on  $\nu^{-1}$  is not an intrinsic property of the turbulent dynamics. Rather, it appears to be a “removable artifact” brought about by the particular form of the forcing term used as a model for energy and enstrophy injection that drives the turbulence. Indeed, the “extra” dependence of  $D_H$  on  $\nu^{-1}$  would be removed if the energy and enstrophy injection could be made viscosity independent (more precisely if the injection could be bounded independently of viscosity) provided that this forcing model does not jeopardize the existence of the global attractor.

**II. PRELIMINARIES**

In this section we briefly recall a recently derived upper bound [7] for the enstrophy dissipation wave number  $k_d$ . We deduce from this result an estimate for the number of active modes, by counting all modes having length scales larger than the dissipation length scale corresponding to  $k_d$ . A brief functional setting of the two-dimensional Navier-Stokes system in the vorticity and stream function formulation is described, and the problem of phase-space volume evolution is formulated. We avoid technical detail and use informal language.

**A. Number of active modes**

The two-dimensional Navier-Stokes system written in terms of the stream function  $\psi$  and vorticity  $\omega = \Delta\psi$  is

$$\omega_t + J(\psi, \omega) = \nu \Delta \omega, \tag{2}$$

where  $J(\psi, \omega) = \psi_x \omega_y - \psi_y \omega_x$  and  $\nu$  is the viscosity. We consider Eq. (2) in a doubly periodic domain of size  $2\pi L$ . The initial vorticity field  $\omega_0$  is assumed to be smooth and has zero average. Equation (2) preserves the zero-mean property. This, together with periodicity, allows  $\omega$  (and  $\psi$ ) to be expressible as a Fourier series in terms of  $\sin L^{-1}(\ell x + m y)$  and  $\cos L^{-1}(\ell x + m y)$ , where  $\ell$  and  $m$  are integers not simultaneously zero. In other words, the infinite-dimensional space (solution function space) can be spanned by the infinite basis  $\{\sin L^{-1}(\ell x + m y), \cos L^{-1}(\ell x + m y)\}$ . Here  $(\ell, m)$  can be identified with a lattice of unit spacing on the upper half plane, with points on either half of the horizontal axis—including the origin—removed.

The advection term  $J(\psi, \omega)$  has many conservation laws. In particular, the total kinetic energy  $\|\nabla\psi\|^2/2 = \langle |\nabla\psi|^2 \rangle/2 = \int |\nabla\psi|^2 dx dy/2$ , the total enstrophy  $\|\omega\|^2/2$ , and the peak vorticity  $\|\omega\|_\infty$  are conserved. These are the most important conserved quantities and play prominent roles in the theory of turbulence. Under viscous effects, all these quantities decay, though in general at different rates. The enstrophy decays most rapidly, while the kinetic energy and the peak vorticity are far better conserved, with the latter probably best conserved [8].

For relatively small  $\nu$ , the free decay of a general smooth vorticity field presumably becomes turbulent, featuring a wide range of dynamically interacting scales that extend to the viscous dissipation range. This range is characterized by the dissipation wave number  $k_\nu$ , which, according to the phenomenological theory of turbulence [9], is given by  $k_\nu = \chi^{1/6}/\nu^{1/2}$ . Here  $\chi = \nu \|\nabla\omega\|^2/(4\pi^2 L^2)$  denotes the enstrophy dissipation rate per unit area. Recently, Tran [7] derived the upper bound

$$\nu \|\nabla\omega\|^2 \leq \|\omega\|_\infty \|\omega\|^2, \tag{3}$$

for the dissipation rate  $\nu \|\nabla\omega\|^2$  at its peak. Since both vorticity norms on the right-hand side of Eq. (3) decay we have the bound

$$\|\nabla\omega\| \leq \frac{\|\omega_0\|_\infty^{1/2} \|\omega_0\|}{\nu^{1/2}}, \tag{4}$$

which is valid uniformly in time, and the bound

$$k_d = \frac{\|\nabla\omega\|}{\|\omega\|} \leq \frac{\|\omega_0\|_\infty^{1/2}}{\nu^{1/2}}, \tag{5}$$

which is valid at least up to (and probably beyond) the time of peak enstrophy dissipation. The bound for the newly defined enstrophy dissipation wave number  $k_d$  compares favorably to  $k_\nu$ , as it could be significantly smaller than  $k_\nu$  [7]. By the very definition (5), enstrophy dissipation is strongest in the vicinity of  $k_d$ . The wave numbers greater than  $k_d$  are effectively suppressed by viscous forces and virtually inactive. The number of dynamically active modes  $N_c$  corresponding to  $k \leq k_d$  are therefore given by

$$N_c \approx \frac{k_d^2}{k_0^2} \leq \frac{L^2 \|\omega_0\|_\infty}{\nu}, \tag{6}$$

where  $k_0 = 1/L$  is the smallest wave number. The quantity  $L\|\omega_0\|_\infty$  may be identified with the fluid velocity. Perhaps,  $\|\omega_0\|$  is a better representative of the fluid velocity; nevertheless, when it comes to the definition of the Reynolds number  $Re$ , we use  $L\|\omega_0\|_\infty$  and  $\|\omega_0\|$  interchangeably. With this identification, the term on the right-hand side of Eq. (6) may be defined as the Reynolds number  $Re$ . Hence, Eq. (6) can be rewritten in a more compact form

$$N_c \leq Re. \tag{7}$$

From our experience in numerical simulations of two-dimensional turbulence, the estimate (7) is sharp—in fact, spot on. For example, for the standard numerical domain  $2\pi \times 2\pi$  and an initial vorticity maximum  $\|\omega_0\|_\infty \approx 4\pi$ , the simulations of Dritschel *et al.* [8] using  $4\pi(8/3)^2/\nu$  grid points adequately resolved the dissipation scales. This resolution is obviously consistent with Eq. (7), within 1 order of magnitude. As will be seen in the next section, the estimate (7) for  $N_c$  fully agrees with the number of degrees of freedom discussed above.

**B. Problem formulation**

The problem of phase-space volume element contraction (or expansion) is intimately related to the stability of solution with respect to disturbances. To investigate this problem, we consider the linear evolution of a deviation  $\phi$  of the stream function  $\psi$  (corresponding to a deviation  $\Delta\phi$  of the vorticity  $\omega$ ) governed by the linearized equation

$$\Delta\phi_t + J(\phi, \omega) + J(\psi, \Delta\phi) = \nu \Delta^2 \phi, \tag{8}$$

where  $\omega$  (and  $\psi$ ) solves Eq. (2) with initial vorticity  $\omega_0$  (and initial stream function  $\psi_0$ ). By taking the scalar product  $\langle \cdot, \cdot \rangle$  of Eq. (8) with  $\phi$  and  $\Delta\phi$ , we obtain the respective evolution equations for the energy norm  $\|\nabla\phi\|$  and enstrophy norm  $\|\Delta\phi\|$ ,

$$\|\nabla\phi\| \frac{d}{dt} \|\nabla\phi\| = \langle \phi, J(\psi, \Delta\phi) \rangle - \nu \|\Delta\phi\|^2 \tag{9}$$

and

$$\|\Delta\phi\| \frac{d}{dt} \|\Delta\phi\| = -\langle \Delta\phi J(\phi, \omega) \rangle - \nu \|\nabla\Delta\phi\|^2. \quad (10)$$

The respective exponential growth (or decay) rates  $\lambda$  and  $\Lambda$  for  $\|\nabla\phi\|$  and  $\|\Delta\phi\|$  can be readily deduced and are given by

$$\lambda = \frac{d}{dt} \ln \|\nabla\phi\| = \frac{1}{\|\nabla\phi\|^2} [\langle \phi J(\psi, \Delta\phi) \rangle - \nu \|\Delta\phi\|^2] \quad (11)$$

and

$$\Lambda = \frac{d}{dt} \ln \|\Delta\phi\| = \frac{-1}{\|\Delta\phi\|^2} [\langle \Delta\phi J(\phi, \omega) \rangle + \nu \|\nabla\Delta\phi\|^2]. \quad (12)$$

These rates provide a comprehensive picture of solution stability, quantitatively describing how solutions with nearby initial data disperse from one another.

Two natural norms for the present problem are the energy and enstrophy norms. We will refer to the phase space equipped with the energy (enstrophy) norm as the energy (enstrophy) space. In the course of evolution, consider a trajectory commencing from a given initial condition. At an arbitrary point on the trajectory (i.e., at an arbitrary instance in time  $t > 0$ ), we calculate the greatest growth rate  $\lambda$  ( $\Lambda$ ) and identify the corresponding most unstable “direction” by considering the problem of maximizing  $\lambda$  ( $\Lambda$ ) with respect to all admissible  $\phi$ . We denote by  $(\lambda_1, \varphi^1)$   $(\Lambda_1, \vartheta^1)$  the solution of this problem, where for convenience  $\varphi^1$  ( $\vartheta^1$ ) has been normalized, i.e.,  $\|\nabla\varphi^1\|=1$  ( $\|\Delta\vartheta^1\|=1$ ). The second greatest rate  $\lambda_2$  ( $\Lambda_2$ ) and the corresponding second most unstable direction  $\varphi^2$  ( $\vartheta^2$ ) orthogonal to  $\varphi^1$  ( $\vartheta^1$ ) is obtained by the same maximization problem subject to the orthogonality constraint, i.e.,  $\langle \nabla\varphi^1 \cdot \nabla\varphi^2 \rangle = 0$  ( $\langle \Delta\vartheta^1 \Delta\vartheta^2 \rangle = 0$ ). By repeating this procedure  $n$  times, we obtain the set  $\{\varphi^1, \varphi^2, \dots, \varphi^n\}$  ( $\{\vartheta^1, \vartheta^2, \dots, \vartheta^n\}$ ) of mutually orthonormal functions and the corresponding set of ordered rates  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  ( $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_n$ ). These may be defined as the first  $n$  local Lyapunov exponents, and their existence is guaranteed since the maximization problems are expected to return unique solutions. Note that for the conventional Lyapunov exponents, existence can be a major issue, even for low-dimensional systems of a few degrees of freedom.

Now, in the linear subspace spanned by  $\{\varphi^1, \varphi^2, \dots, \varphi^n\}$  ( $\{\vartheta^1, \vartheta^2, \dots, \vartheta^n\}$ ), consider an  $n$ -dimensional ball  $B(\cdot, r)$  of radius  $r$  centered at the point discussed above. The  $n$ -dimensional volumes  $v$  (in the energy subspace) and  $V$  (in the enstrophy subspace) of  $B(\cdot, r)$  are given by  $v \propto r^n \|\nabla\varphi^1\| \|\nabla\varphi^2\| \dots \|\nabla\varphi^n\| = r^n$  and  $V \propto r^n \|\Delta\vartheta^1\| \|\Delta\vartheta^2\| \dots \|\Delta\vartheta^n\| = r^n$ , respectively. (See the book of Temam [10] for a formal definition of volume based on the related concept of exterior product.) The respective equations governing the evolution of  $v$  and  $V$  under the linearized dynamics described by Eq. (8) are

$$\frac{d}{dt} \ln v = \sum_{i=1}^n \lambda_i = \sum_{i=1}^n [\langle \varphi^i J(\psi, \Delta\varphi^i) \rangle - \nu \|\Delta\varphi^i\|^2] \quad (13)$$

and

$$\frac{d}{dt} \ln V = \sum_{i=1}^n \Lambda_i = - \sum_{i=1}^n [\langle \Delta\vartheta^i J(\vartheta^i, \omega) \rangle + \nu \|\nabla\Delta\vartheta^i\|^2]. \quad (14)$$

In deriving Eqs. (13) and (14), we have used Eqs. (11) and (12), respectively. The sum  $\sum_{i=1}^n \lambda_i$  ( $\sum_{i=1}^n \Lambda_i$ ) represents the exponential growth or decay rate of  $v$  ( $V$ ). When this sum is negative, the volume of the  $n$ -dimensional ball  $B(\cdot, r)$  contracts exponentially. Note that by construction,  $B(\cdot, r)$  is optimally “oriented” to be least contracting. This means that if  $\sum_{i=1}^n \lambda_i$  ( $\sum_{i=1}^n \Lambda_i$ ) is negative, then volume contraction becomes universal for all  $n$ - or higher-dimensional balls locally centered at the point in question. Furthermore, if this point is taken arbitrarily on the trajectory, which is the case in this study, then volume contraction becomes universal along the trajectory.

The determination of  $N$  then reduces to minimizing  $n$  such that the sum on the right-hand side of Eqs. (13) and (14) is negative. We use the mathematical techniques developed in the 1980s by Babin and Vishik [4] and Constantin *et al.* [5,6,11] for estimating the attractor dimension of forced two-dimensional Navier-Stokes turbulence. See also the paper of Doering and Gibbon [12] for the same treatment in the stream function and vorticity setting. As can be seen in the next section, the derivation of upper bounds for  $N$  is equivalent to the determination of the Hausdorff dimension of the global attractor in the forced case. The main difference is that although the present formulation is specifically designed to handle the decaying case, which has a trivial global attractor, its scope of application is broad. In general, the present notion of degrees of freedom makes sense for general dissipative dynamical systems provided that bounded solutions exist. There are virtually no other technical requirements for the application of the method. In particular, no *a priori* knowledge of the existence of an attractor is required.

### III. RESULTS

This section presents the calculations described above, leading to upper bounds for  $N$ . The treatment is relatively self-contained. However, the reader, who is interested in further detail related to the analytic inequalities employed in various stages of the calculations, is referred to the cited papers and references therein.

#### A. Degrees of freedom in energy space

We begin by deriving an upper bound for  $N$  in the energy space. From Eq. (13) we have

$$\begin{aligned}
 \sum_{i=1}^n \lambda_i &= \sum_{i=1}^n [\langle \varphi^i J(\psi, \Delta \varphi^i) \rangle - \nu \|\Delta \varphi^i\|^2] = - \sum_{i=1}^n [\langle \Delta \varphi^i J(\psi, \varphi^i) \rangle + \nu \|\Delta \varphi^i\|^2] = - \sum_{i=1}^n [\langle \varphi^i J(\psi_x, \varphi_x^i) + \varphi^i J(\psi_y, \varphi_y^i) \rangle + \nu \|\Delta \varphi^i\|^2] \\
 &= \sum_{i=1}^n [\langle \varphi_x^i J(\psi_x, \varphi^i) + \varphi_y^i J(\psi_y, \varphi^i) \rangle - \nu \|\Delta \varphi^i\|^2] \leq \sum_{i=1}^n [\langle |\nabla \varphi^i| (|\varphi_x^i| |\nabla \psi_x| + |\varphi_y^i| |\nabla \psi_y|) \rangle - \nu \|\Delta \varphi^i\|^2] \\
 &\leq \sum_{i=1}^n [\langle |\nabla \varphi^i|^2 (|\nabla \psi_x|^2 + |\nabla \psi_y|^2)^{1/2} \rangle - \nu \|\Delta \varphi^i\|^2] \leq \|\omega\| \left\| \sum_{i=1}^n |\nabla \varphi^i|^2 \right\| - \nu \sum_{i=1}^n \|\Delta \varphi^i\|^2, \tag{15}
 \end{aligned}$$

where integration by parts and the Cauchy-Schwarz inequality have been used. For further estimates of the terms on the right-hand side of Eq. (15), we employ the following two analytic inequalities concerning the orthonormal set  $\{\varphi^i\}_{i=1}^n$  with respect to the energy norm. First, we have the Lieb-Thirring inequality [6,10]

$$\left\| \sum_{i=1}^n |\nabla \varphi^i|^2 \right\| \leq c_1 \left( \sum_{i=1}^n \|\Delta \varphi^i\|^2 \right)^{1/2}, \tag{16}$$

where  $c_1$  is a nondimensional constant independent of the set  $\{\varphi^i\}_{i=1}^n$ . Second, we know that for  $n \gg 1$ , there are approximately  $n$  basis functions (trigonometric functions mentioned earlier) within the wave-number radius  $\sqrt{n}/L$ . Their (repeated) eigenvalues under  $-\Delta$  are  $(\ell^2 + m^2)/L^2$ , where  $\ell^2 + m^2 \leq n$ . These constitute the first  $n$  eigenvalues (in nondecreasing order) of  $-\Delta$  and sum up to approximately  $n^2/L^2$ . It follows from the Rayleigh-Ritz principle that

$$\sum_{i=1}^n \|\Delta \varphi^i\|^2 \geq \frac{c_2^2}{L^2} n^2, \tag{17}$$

where  $c_2$  is another nondimensional constant independent of the set  $\{\varphi^i\}_{i=1}^n$ . By substituting Eqs. (16) and (17) into Eq. (15), we obtain

$$\sum_{i=1}^n \lambda_i \leq \left( \sum_{i=1}^n \|\Delta \varphi^i\|^2 \right)^{1/2} \left( c_1 \|\omega\| - \nu \frac{c_2}{L} n \right). \tag{18}$$

It follows that  $\sum_{i=1}^n \lambda_i \leq 0$  when  $n \geq c_1 L \|\omega\| / (c_2 \nu)$ . Hence we deduce the bound

$$N \leq C_1 \frac{L \|\omega\|}{\nu} \leq C_1 \frac{L \|\omega_0\|}{\nu} = C_1 \text{Re}, \tag{19}$$

where  $C_1 = c_1/c_2$  and  $\text{Re}$  has been redefined by replacing  $L \|\omega_0\|_\infty$  with  $\|\omega_0\|$ . Note that the precise result should be that  $N$  is no greater than the least integral upper bound for  $C_1 \text{Re}$ ; however, in writing Eq. (19), we have opted to ignore this exceedingly minor detail. Equation (19) gives a clear linear dependence of  $N$  on  $\text{Re}$ . Thus, we have essentially recovered the bound (7), up to the constant factor  $C_1$  and a slight difference in the definition of  $\text{Re}$ , which was obtained earlier by counting the active modes from the smallest wave number  $k_0 = 1/L$  to the dissipation wave number  $k_d = \|\nabla \omega\|/\|\omega\|$ .

## B. Degrees of freedom in enstrophy space

An upper bound for  $N$  in the enstrophy space is derived in a similar manner. From Eq. (14) we have

$$\begin{aligned}
 \sum_{i=1}^n \Lambda_i &= - \sum_{i=1}^n [\langle \Delta \vartheta^i J(\vartheta^i, \omega) \rangle + \nu \|\nabla \Delta \vartheta^i\|^2] \\
 &\leq \sum_{i=1}^n (\langle |\Delta \vartheta^i| |\nabla \vartheta^i| |\nabla \omega| \rangle - \nu \|\nabla \Delta \vartheta^i\|^2) \\
 &\leq \left\langle \left( \sum_{i=1}^n |\Delta \vartheta^i|^2 \sum_{i=1}^n |\nabla \vartheta^i|^2 \right)^{1/2} |\nabla \omega| \right\rangle - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2 \\
 &\leq \left\langle \left( \sum_{i=1}^n |\Delta \vartheta^i|^2 \sum_{i=1}^n |\nabla \vartheta^i|^2 \right)^2 \right\rangle^{1/4} \langle |\nabla \omega|^{4/3} \rangle^{3/4} \\
 &\quad - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2 \\
 &\leq \left\| \sum_{i=1}^n |\nabla \vartheta^i|^2 \right\|_\infty^{1/2} \left\| \sum_{i=1}^n |\Delta \vartheta^i|^2 \right\|^{1/2} (2\pi L)^{1/2} \|\nabla \omega\| \\
 &\quad - \nu \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2, \tag{20}
 \end{aligned}$$

where Hölder's inequalities with the pairs of conjugate exponents  $4/3$  and  $4$  and  $3/2$  and  $3$  have been used in the penultimate and final steps, respectively. For further estimates of the terms in this equation, we employ a few more analytic inequalities concerning the orthonormal set  $\{\vartheta^i\}_{i=1}^n$  in the enstrophy space. First, we have [6,11]

$$\left\| \sum_{i=1}^n |\nabla \vartheta^i|^2 \right\|_\infty \leq c_3^2 \left( 1 + \ln \sum_{i=1}^n L^2 \|\nabla \Delta \vartheta^i\|^2 \right), \tag{21}$$

where  $c_3$  is a nondimensional constant independent of the set  $\{\vartheta^i\}_{i=1}^n$ . Second, a version of Eq. (16) for the present orthonormal set is

$$\left\| \sum_{i=1}^n |\Delta \vartheta^i|^2 \right\| \leq c_1 \left( \sum_{i=1}^n \|\nabla \Delta \vartheta^i\|^2 \right)^{1/2}. \tag{22}$$

Finally, a version of Eq. (17) for  $\{\vartheta^i\}_{i=1}^n$  is

$$\sum_{i=1}^n \|\nabla\Delta\vartheta^i\|^2 \geq \frac{c_2^2}{L^2} n^2. \quad (23)$$

Now by substituting Eqs. (4), (21), and (22) into Eq. (20) we obtain

$$\begin{aligned} \sum_{i=1}^n \Lambda_i &\leq C' \left( 1 + \ln \sum_{i=1}^n L^2 \|\nabla\Delta\vartheta^i\|^2 \right)^{1/2} \\ &\times \left( \sum_{i=1}^n L^2 \|\nabla\Delta\vartheta^i\|^2 \right)^{1/4} \frac{\|\omega_0\|_{\infty}^{1/2} \|\omega_0\|}{\nu^{1/2}} - \nu \sum_{i=1}^n \|\nabla\Delta\vartheta^i\|^2 \\ &= \frac{\nu \xi^{1/4}}{L^2} \left[ C' (1 + \ln \xi)^{1/2} \frac{L^2 \|\omega_0\|_{\infty}^{1/2} \|\omega_0\|}{\nu^{3/2}} - \xi^{3/4} \right] \\ &= \frac{\nu \xi^{1/4}}{L^2} [C' \text{Re}^{3/2} (1 + \ln \xi)^{1/2} - \xi^{3/4}], \end{aligned} \quad (24)$$

where  $C' = \sqrt{2\pi} c_1 c_3$ ,  $\xi = \sum_{i=1}^n L^2 \|\nabla\Delta\vartheta^i\|^2$ , and  $\text{Re} = (L^4 \|\omega_0\|_{\infty} \|\omega_0\|^2)^{1/3} / \nu$ . Note that by Eq. (23) we have  $\xi \geq c_2^2 n^2$ . Hence without the logarithmic term, it would be straightforward to substitute this into Eq. (24) and deduce an upper bound for  $N$  similar to Eq. (19) with the newly defined  $\text{Re}$  replacing its previously defined (and comparable) counterpart. Since we are interested in the case  $\xi \gg 1$ , the logarithmic term should introduce a small departure to the linear dependence of  $N$  on  $\text{Re}$  only. In order to account for  $\ln \xi$ , we can “cover” it by a fraction of  $\xi$ , say  $\xi/2$ . By elementary calculus, we find that

$$C' \text{Re}^{3/2} (1 + \ln \xi)^{1/2} - \frac{\xi^{3/4}}{2} \leq \sqrt{2} C' \text{Re}^{3/2} (1 + \ln \text{Re})^{1/2}, \quad (25)$$

where we have dropped a negative term on the right-hand side. It follows that

$$\begin{aligned} C' \text{Re}^{3/2} (1 + \ln \xi)^{1/2} - \xi^{3/4} &\leq \sqrt{2} C' \text{Re}^{3/2} (1 + \ln \text{Re})^{1/2} - \frac{\xi^{3/4}}{2} \\ &\leq \sqrt{2} C' \text{Re}^{3/2} (1 + \ln \text{Re})^{1/2} - \frac{(c_2 n)^{3/2}}{2}. \end{aligned} \quad (26)$$

The condition  $\sum_{i=1}^n \Lambda_i \leq 0$  is satisfied when the right-hand side of Eq. (26) is nonpositive. This requires a straightforward condition for  $n$  which in turn yields the result

$$N \leq C_2 \text{Re} (1 + \ln \text{Re})^{1/3}, \quad (27)$$

where  $C_2 = (8C'^2)^{1/3} / c_2$ .

As expected, Eq. (27) gives an essentially linear scaling of  $N$  with  $\text{Re}$  since the superlinear dependence on  $\text{Re}$ , due to

the logarithmic term, is slight for large  $\text{Re}$ . Given that the same linear scaling was found earlier in the energy space, this is somewhat surprising. The reason is that the energy in two-dimensional turbulence is predominantly transferred to smaller wave numbers while the enstrophy is predominantly transferred to larger wave numbers. This undoubtedly implies that the enstrophy dynamics have relatively more degrees of freedom than the energy dynamics. Hence, it is somewhat counter intuitive that Eqs. (19) and (27) do not differ by much. A possible explanation is that Eq. (19) may not be as optimal as Eq. (27). Some qualitative support for this possibility turns up in the next subsection.

### C. Discussion

In the 1980s, estimates were derived for the Hausdorff dimension  $D_H$  of the global attractor of the two-dimensional Navier-Stokes system driven by a time-independent force  $f$  [4–6]. These estimates have been known to be sharp, allowing just minor improvements for the attractor dimension in the energy space only [13,14]. In the present notations, the respective bounds for  $D_H$  in the energy and enstrophy spaces are given by

$$D_H \leq c' \frac{L \|\nabla^{-1} f\|}{\nu^2} \leq c' \frac{L^2 \|f\|}{\nu^2} = c' G \quad (28)$$

and

$$D_H \leq c'' \left( \frac{L^2 \|f\|}{\nu^2} \right)^{2/3} \left( 1 + \ln \frac{L^2 \|f\|}{\nu^2} \right)^{1/3} = c'' G^{2/3} (1 + \ln G)^{1/3}, \quad (29)$$

where  $c'$  and  $c''$  are constant and  $G$  is known as the generalized Grasshof number. Although  $G$  has some certain physical significance, its highly superlinear dependence on  $\nu^{-1}$  appears to make Eqs. (28) and (29) in disagreement with the bounds for  $N_c$  and for  $N$  derived earlier. We claim that this apparent disagreement is due entirely to the particular form of  $f$  and could be fully reconciled. For the remainder of this paper, we will elaborate on this claim.

Due to rigor requirement in the mathematical formulation [4–6], a time-independent forcing  $f$  has been used as a model for energy and enstrophy injection. The forced Navier-Stokes equations

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \quad (30)$$

then admit the following evolution equations:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 = -\nu \|\nabla \mathbf{u}\|^2 + \langle \mathbf{u} \cdot \mathbf{f} \rangle \leq -\nu \|\nabla \mathbf{u}\|^2 + \|\nabla \mathbf{u}\| \|\nabla^{-1} f\| \leq -\frac{\nu}{2} \|\nabla \mathbf{u}\|^2 + \frac{\|\nabla^{-1} f\|^2}{2\nu} \quad (31)$$

and

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|^2 = -\nu \|\Delta \mathbf{u}\|^2 - \langle \Delta \mathbf{u} \cdot \mathbf{f} \rangle \leq -\nu \|\Delta \mathbf{u}\|^2 + \|\Delta \mathbf{u}\| \|\mathbf{f}\| \leq -\frac{\nu}{2} \|\Delta \mathbf{u}\|^2 + \frac{\|\mathbf{f}\|^2}{2\nu} \quad (32)$$

for the energy  $\|\mathbf{u}\|^2/2$  and enstrophy  $\|\nabla \mathbf{u}\|^2/2$ , respectively. In Eqs. (31) and (32), the terms  $\|\nabla^{-1} \mathbf{f}\|^2/(2\nu)$  and  $\|\mathbf{f}\|^2/(2\nu)$  represent upper bounds for the energy and enstrophy injection rates, respectively. Their dependence on  $\nu$  is inescapable because the injection rates  $\langle \mathbf{u} \cdot \mathbf{f} \rangle$  and  $-\langle \Delta \mathbf{u} \cdot \mathbf{f} \rangle$  themselves are flow dependent.

We now demonstrate how the viscosity dependence of the injection rates (or more precisely of the upper bounds for the injection rates) contributes to the superlinear scaling of  $D_H$  with  $\nu^{-1}$ . To this end, let us recall the intermediate steps [5,6,10] toward Eqs. (28) and (29) given below

$$D_H \leq c' \frac{L \overline{\|\nabla \mathbf{u}\|^2}^{1/2}}{\nu} \quad (33)$$

and

$$D_H \leq c'' \left( \frac{L^2 \overline{\|\Delta \mathbf{u}\|^2}^{1/2}}{\nu} \right)^{2/3} \left( 1 + \ln \frac{L^2 \overline{\|\Delta \mathbf{u}\|^2}^{1/2}}{\nu} \right)^{1/3}, \quad (34)$$

where the overline denotes the supremum of an asymptotic average. From Eqs. (31) and (32) we can deduce the forced dissipative balance equations

$$\overline{\|\nabla \mathbf{u}\|^2}^{1/2} \leq \frac{\|\nabla^{-1} \mathbf{f}\|}{\nu} \quad (35)$$

and

$$\overline{\|\Delta \mathbf{u}\|^2}^{1/2} \leq \frac{\|\mathbf{f}\|}{\nu}. \quad (36)$$

Upon substituting these into Eqs. (33) and (34), we recover Eqs. (28) and (29), respectively.

However, if the driving force could somehow be modeled in such a way that the averaged energy and enstrophy injection rates would be bounded independently of viscosity, say by  $\epsilon^2$  and  $\eta^2$ , respectively, then Eqs. (35) and (36) would become

$$\overline{\|\nabla \mathbf{u}\|^2}^{1/2} \leq \frac{\epsilon}{\nu^{1/2}} \quad (37)$$

and

$$\overline{\|\Delta \mathbf{u}\|^2}^{1/2} \leq \frac{\eta}{\nu^{1/2}}. \quad (38)$$

Then upon substituting these into Eqs. (33) and (34), we obtain

$$D_H \leq c' \frac{L\epsilon}{\nu^{3/2}} \quad (39)$$

and

$$D_H \leq c'' \left( \frac{L^2 \eta}{\nu^{3/2}} \right)^{2/3} \left( 1 + \ln \frac{L^2 \eta}{\nu^{3/2}} \right)^{1/3}. \quad (40)$$

One can see that Eq. (40) has the desired scaling, i.e., linear dependence on  $\nu^{-1}$  with a logarithmic “correction” as in Eq. (27). Hence, for the dimension estimate in the enstrophy space, the “extra” dependence on  $\nu^{-1}$  would be completely removed if the enstrophy injection could be bounded independently of viscosity. Note, however, that only part of the extra dependence on  $\nu^{-1}$  would be removed from the dimension estimate in the energy space when the energy injection is made independent of viscosity. This strengthens our earlier suggestion that the estimate for  $N$  (and  $D_H$ ) in the energy space might not be as optimal as its counterpart in the enstrophy space.

Closely related to the present notion of degrees of freedom are the concepts of determining modes, nodes, and finite-volume elements [15–19]. A large body of research on the number of degrees of freedom deduced from these concepts has produced upper bounds proportional to  $G$  (cf. [19]). Like the estimate for  $D_H$  in the energy space, these bounds would reduce to  $\propto \nu^{-3/2}$  if the enstrophy injection could be bounded independently of viscosity. In another mathematical study [20] relevant to the present problem, it has been found that when the time-independent  $\mathbf{f}$  in Eq. (30) consists of a single Fourier mode, the unstable manifold emanating from the stationary solution  $-\Delta^{-1} \mathbf{f}/\nu$  has a dimension not lower than  $\propto G^{2/3}$ . This is a lower bound for the Hausdorff dimension  $D_H$  of the global attractor, and the superlinear dependence on  $\nu^{-1}$  of this bound is a consequence of the dependence on  $\nu$  of the stationary solution.

In passing, it is worth mentioning that numerical simulations of two-dimensional turbulence have routinely used a variety of forcing that provides steady energy and enstrophy injection rates  $\epsilon^2$  and  $\eta^2$ . This class of forcing includes white-noise and flow-dependent forcing [21,22]. While such a class of forcing is numerically desirable and realistic in some sense, it may render Eq. (30) incompatible with the mathematical formulation leading to the desired estimate (40). Nevertheless, for the present approach, there are no technical difficulties in arriving at this estimate as an upper bound for the number of degrees of freedom in the present sense.

#### IV. CONCLUSION

In conclusion, we have derived upper bounds for the number of degrees of freedom  $N$  of two-dimensional Navier-Stokes turbulence freely evolving from a smooth initial vorticity field in a doubly periodic domain. This number is defined as the minimum dimension such that arbitrary phase-space volume elements of no lower dimensions along the solution curve in phase-space contract exponentially under

the linearized dynamics. This means that the (locally in time) turbulent dynamics could be sufficiently “contained” within a linear subspace whose dimension does not exceed  $N$ . In essence,  $N$  represents a reduced dimension that a modeled system should achieve in order to describe the turbulence adequately. It is found that  $N \leq C_1 \text{Re}$  in the energy space and  $N \leq C_2 \text{Re}(1 + \ln \text{Re})^{1/3}$  in the enstrophy space. Here  $C_1$  and  $C_2$  are constant and  $\text{Re}$  is the Reynolds number, which is defined in terms of the initial vorticity, the system size, and the viscosity. These results are consistent with the number of active modes deduced from a recent mathematical estimate of the viscous dissipation wave number  $k_d = \|\nabla \omega\|/\|\omega\|$ .

The present estimates for  $N$  have been compared with well-known bounds for the Hausdorff dimension  $D_H$  of the global attractor in the forced case, and the apparent difference between the linear (or nearly so) scaling of  $N$  with  $\text{Re}$  and the highly superlinear dependence of  $D_H$  on the inverse viscosity  $\nu^{-1}$  has been discussed. We have argued that the superlinear dependence of  $D_H$  on  $\nu^{-1}$  is not an intrinsic property of the turbulent dynamics and further suggested that this is a “removable artifact,” arising from the use of a time-independent forcing as a model for energy and enstrophy injection that drives the turbulence. This suggestion has been strengthened by the fact that the “extra” dependence of  $D_H$  on  $\nu^{-1}$  would be completely removed (at least for the estimate of  $D_H$  in the enstrophy space) if one could model the driving force in such a way that the enstrophy injection rate does not depend on the viscosity. Such a forcing can be seen to be more realistic than ones with viscosity-dependent input.

In the present analysis, we simply follow a trajectory starting from an arbitrary smooth initial vorticity field in the

solution (function) space of the two-dimensional Navier-Stokes equations and monitor the evolution (under the linearized dynamics) of the volumes of  $n$ -dimensional balls centered on the trajectory. We estimate how large  $n$  should be to ensure that these volumes contract exponentially. This turns out to be equivalent to the method of estimating the Hausdorff dimension of the global attractor of the forced system. The present approach can be seen to be highly flexible in application. In general, it is applicable to either autonomous or nonautonomous, forced or unforced, and finite-dimensional or infinite-dimensional systems. There are virtually no special requirements, other than existence of solution, for the present definition (and method of analysis) of the number of degrees of freedom to make sense. In particular, the existence of the usual Lyapunov exponents is not an issue. Furthermore, there is no need for *a priori* knowledge of the existence of an attractor (or a global attractor), whose generalized dimensions would normally be considered as the number of degrees of freedom of the dynamical system in question. Given all this, we may apply the present approach to less idealized and more realistic dynamical models without risking to compromise mathematical rigor.

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